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The least-damped disturbance to Poiseuille flow in a circular pipe

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The properties of infinitesimal disturbances to Poiseuille flow in a circular pipe have been found for a wide range of wavenumbers through recent numerical work (Salwen & Grosch 1972; Garg & Rouleau 1972). These studies did not, however, find the *least*-damped disturbances. In this paper, the properties of disturbances are found in a limiting case. These disturbances are thought to have decay rates which are equal to or very close to the smallest value possible for any given large value of the Reynolds number R. For disturbances which decay in *time*, the limiting disturbances can be found analytically. They have the property that the axial wavenumber α tends to zero as $R \to \infty$. The smallest decay rate $-\beta_i$ is given by

$$-\beta_i R = j_{1,1}^2 \approx 14{\cdot}7,$$

where $j_{1,1}$ is the first zero of the Bessel function J_1. Two modes have this decay rate. One is axisymmetric with motion only in the azimuthal direction, and the other has azimuthal wavenumber $n = 1$. For disturbances which decay in *space*, the limiting solutions can be found by numerically evaluating power series. They have the property that the frequency β tends to zero as R tends to infinity. The smallest decay rate α_i for these disturbances is given by

$$\alpha_i R \approx 21{\cdot}4,$$

corresponding to an axisymmetric mode with motion only in the azimuthal direction. A mode with azimuthal wavenumber $n = 1$ has a slightly larger decay rate given by

$$\alpha_i R \approx 28{\cdot}7.$$

Part 1. Temporally damped disturbances

1. Introduction

The problem of the stability of Poiseuille flow in a circular pipe has a long history. Experiments by Ekman (1910) and others have shown that, by taking sufficient care with entry conditions, the onset of instability can be delayed to very high Reynolds numbers (40 000 and more), indicating that Poiseuille flow in a circular pipe is stable to infinitesimal disturbances but unstable to disturbances of finite amplitude. Stability to infinitesimal disturbances has not been proved rigorously, but the properties of infinitesimal disturbances have now been studied over a wide range of conditions (see Salwen & Grosch 1972), and all found to decay [Graebel (1970) found indications to the contrary, but his results are not supported by other work, such as that of Burridge (1969) and Salwen & Grosch (1972)].

Salwen & Grosch (1972) undertook very extensive numerical calculations for disturbances whose amplitude changes only with time. They explored the stability characteristics of these disturbances for azimuthal wavenumbers $n = 0, 1, 2, 3, 4$ and 5, for axial wavenumbers α between $0{\cdot}1$ and 10 and for values of αR up to 50000, R being the Reynolds number. For a given mode with fixed α and n, one of two characteristic behaviours is found as $R \rightarrow \infty$. Either

(i) the wave speed c_r tends to unity and the decay rate $-\beta_i$ satisfies

$$-\beta_i R \rightarrow B_m(\alpha R)^{\frac{1}{2}} \quad \text{as} \quad \alpha R \rightarrow \infty, \tag{1.1}$$

where B_m is a constant, or

(ii) the wave speed c_r tends to zero and the decay rate $-\beta_i$ satisfies

$$-\beta_i R \rightarrow F_q(\alpha R)^{\frac{2}{3}} \quad \text{as} \quad \alpha R \rightarrow \infty, \tag{1.2}$$

where F_q is a constant.

In either case, the decay rate tends to zero as $R \rightarrow \infty$, and Salwen & Grosch have determined the coefficients B_m and F_q.

Now consider the dependence of the decay rate $-\beta_i$ on α at a fixed large value of R. For both types (1.1) and (1.2) of mode, the decay rate *decreases* as α decreases until the condition $\alpha R \gg 1$ for the validity of (1.1) and (1.2) is violated. Salwen & Grosch did not calculate solutions for $\alpha < 0{\cdot}1$ and so did not find the *least*-damped disturbance.

The results of this paper are based on earlier experience (Gill 1965) with *spatially* damped *axisymmetric* modes. For a fixed frequency β these modes showed behaviour as $R \rightarrow \infty$ similar to the behaviour given in (i) or (ii) above. Those with wave speed tending to unity (corresponding to (i) above) were called m-modes and those with wave speed tending to zero (corresponding to (ii) above) were called q-modes. For a fixed large value of R, the decay rate of these modes decreased as the frequency β decreased (Gill 1965, figure 5). When βR became of order unity, the character of the modes changed and the decay rate reached its minimum value. The modes were then called l-modes, which are the modes obtained in the limit as $\beta R \rightarrow 0$.

In this part of the paper the equations for the l-modes for temporally decaying disturbances will be derived. The solutions can be obtained analytically and are a special case of solutions given by Burridge & Drazin (1969), who discussed the limit $\alpha R \rightarrow 0$ with α fixed. These solutions were also used by Salwen & Grosch (1972) as expansion functions for their numerical work. However, their significance as least-damped disturbances for *large* Reynolds number has not been pointed out. Here the nature of the limit is discussed and the solutions given in detail for comparison with the spatially decaying solutions which are calculated in part 2.

2. Equations

The equations for infinitesimal disturbances to the basic flow can be put in non-dimensional form by choosing as the respective units of velocity, length and density the maximum velocity U of the basic flow, the radius a of the pipe and the density ρ of the fluid. Then the kinematic viscosity ν of the fluid is replaced in the equations by the reciprocal of the Reynolds number

$$R = Ua/\nu. \tag{2.1}$$

Let (x, r, θ) be cylindrical polar co-ordinates such that $r = 0$ represents the centre-line of the pipe and x increases in the downstream direction, and let t be the time. Then the basic Poiseuille flow has velocity components

$$(1-r^2, 0, 0); \tag{2.2}$$

the perturbation velocity is assumed to have the form

$$[u(r), iv(r), w(r)] \exp(in\theta + i\alpha x - i\beta t) \tag{2.3}$$

and the perturbation pressure to be

$$p(r) \exp(in\theta + i\alpha x - i\beta t). \tag{2.4}$$

The equations satisfied by u, v, w and p are then

$$[\alpha(1-r^2)-\beta]\,u - 2rv = -\alpha p - iR^{-1}\left[u'' + \frac{u'}{r} - \left(\alpha^2 + \frac{n^2}{r^2}\right)u\right], \tag{2.5}$$

$$[\alpha(1-r^2)-\beta]\,v = p' - iR^{-1}\left[v'' + \frac{v'}{r} - \left(\alpha^2 + \frac{n^2+1}{r^2}\right)v - \frac{2n}{r^2}w\right], \tag{2.6}$$

$$[\alpha(1-r^2)-\beta]\,w = -\frac{n}{r}p - iR^{-1}\left[w'' + \frac{w'}{r} - \left(\alpha^2 + \frac{n^2+1}{r^2}\right)w - \frac{2n}{r^2}v\right] \tag{2.7}$$

and

$$\alpha u + v' + \frac{v}{r} + \frac{n}{r}w = 0. \tag{2.8}$$

3. Temporarily damped non-axisymmetric l-modes

The problem for temporally damped disturbances is to find the complex eigenvalues β for (2.5)–(2.8) as functions of the real parameters α, n and R. The l-mode corresponds to the limit in which $R \to \infty$, $\alpha R \to 0$ but for which βR tends to a finite value. The equations satisfied in this limit for $n \neq 0$ are obtained by applying the limiting process to (2.5)–(2.8) keeping αu, v, w and Rp finite. The resulting equations are

$$u'' + \frac{u'}{r} - \frac{n^2}{r^2}u + i\beta R u = 0, \tag{3.1}$$

$$v'' + \frac{v'}{r} - \frac{n^2+1}{r^2}v + i\beta R v - \frac{2n}{r^2}w + iRp' = 0, \tag{3.2}$$

$$w'' + \frac{w'}{r} - \frac{n^2+1}{r^2}w + i\beta R w - \frac{2n}{r^2}v - \frac{in}{r}Rp = 0, \tag{3.3}$$

$$\alpha u + v' + \frac{v}{r} + \frac{n}{r}w = 0. \tag{3.4}$$

The boundary condition at $r = 0$ (see Batchelor & Gill 1962, equations (2.9)–(2.11)) is

$$\left.\begin{aligned} u = p = v+w = 0 \quad &\text{if} \quad n = 1, \\ u = p = v = w = 0 \quad &\text{otherwise.} \end{aligned}\right\} \tag{3.5}$$

There are three solutions satisfying these conditions, namely

$$\text{(i)} \quad \alpha u = 0, \quad v = r^{n-1}, \quad w = -r^{n-1}, \quad np = -\beta r^n, \tag{3.6}$$

$$\text{(ii)} \quad \alpha u = 0, \quad v = nJ_n(\mu r)/\mu r, \quad w = -J_n'(\mu r), \quad p = 0, \tag{3.7}$$

$$\text{(iii)} \quad \alpha u = \mu J_n(\mu r), \quad v = J_n'(\mu r), \quad w = -nJ_n(\mu r)/\mu r, \quad p = 0, \tag{3.8}$$

where

$$\mu^2 = i\beta R. \tag{3.9}$$

From these, there are two linear combinations which satisfy the no-slip condition at $r = 1$ (cf. Burridge & Drazin 1969). The first solution is

$$\left.\begin{aligned} \alpha u &= \mu J_n(\mu r), \\ v+w &= 0, \\ v-w &= 2\,(J_{n-1}(\mu r) - r^{n-1}J_{n-1}(\mu)), \end{aligned}\right\} \tag{3.10}$$

with

$$\mu = j_{n,l} \quad (l = 1, 2, 3, \ldots), \tag{3.11}$$

where $j_{n,l}$ denotes the lth zero of the Bessel function J_n. The least-damped solution of this type occurs for $n = 1$, $l = 1$. From (2.4) the decay rate is $-\beta_i$, where β_i is the imaginary part of β, and in this instance is given by

$$-\beta_i R = j_{1,1}^2 \approx 14{\cdot}7. \tag{3.12}$$

The second solution satisfying the no-slip condition is given by

$$\left.\begin{aligned} \alpha u &= 0, \\ v+w &= J_{n+1}(\mu r), \\ v-w &= J_{n-1}(\mu r) - r^{n-1}J_{n-1}(\mu), \end{aligned}\right\} \tag{3.13}$$

with

$$\mu = j_{n+1,l} \quad (l = 1, 2, 3, \ldots). \tag{3.14}$$

The least-damped mode of this type occurs for $n = 1$, $l = 1$ and has a decay rate $-\beta_i$ given (Sexl 1927, p. 842; Davey & Drazin 1969, p. 216) by

$$-\beta_i R = j_{2,1}^2 \approx 26{\cdot}4. \tag{3.15}$$

This mode decays more quickly than the one found above.

4. Temporally damped axisymmetric *l*-modes

The determination of the limiting solution in the axisymmetric case ($n = 0$) is somewhat different because solution (3.6) is not valid when $n = 0$, and does not satisfy the boundary condition

$$v = w = 0 \quad \text{at} \quad r = 0. \tag{4.1}$$

However, there is a simplification in that (2.7) becomes an equation for w alone while (2.5), (2.6) and (2.8) do not involve w. Thus the equation for w and the equations for u, v and p can be solved independently.

First consider the equation for w. In the limit, this becomes (3.3) with $n = 0$, and the solution is (3.7) with $n = 0$; that is, apart from a constant factor,

$$u = v = p = 0, \quad w = J_1(\mu r). \tag{4.2}$$

The no-slip condition requires

$$\mu = j_{1,l} \quad (l = 1, 2, 3, \ldots), \tag{4.3}$$

a special case of (3.14). The least-damped mode corresponds to $l = 1$, when the damping rate $-\beta_i$ is given by

$$-\beta_i R = j_{1,1}^2 \approx 14{\cdot}7. \tag{4.4}$$

This mode has exactly the same damping rate as the least-damped non-axisymmetric mode, and these modes share the honour of being the temporally damped disturbance with the smallest damping rate.

Now consider the equations for u, v and p. One limiting solution satisfies (3.1), (3.2) and (3.4) and has been found already, being given by (3.8), that is, by

$$\alpha u = \mu J_0(\mu r), \quad v = -J_1(\mu r), \quad w = p = 0. \tag{4.5}$$

The other solution has a different character in that instead of p being of order R^{-1} as before, it is much larger, being of order β/α^2. Equations (2.5), (2.6) and (2.8) now become in the limit

$$-\beta u = -\alpha p, \tag{4.6}$$

$$0 = p', \tag{4.7}$$

$$\alpha u + v' + v/r = 0. \tag{4.8}$$

Note that no viscous terms are involved, so that these are a limiting form of the inviscid equations. The solution is

$$\alpha u = -1, \quad v = \tfrac{1}{2}r, \quad \alpha^2 p = -\beta. \tag{4.9}$$

The linear combination of (4.5) and (4.9) which satisfies the no-slip condition at $r = 1$ is

$$\left.\begin{aligned} \alpha u &= \mu J_0(\mu r) - \mu J_0(\mu), \\ v &= -J_1(\mu r) + \tfrac{1}{2}\mu r J_0(\mu), \end{aligned}\right\} \tag{4.10}$$

where

$$-J_1(\mu) + \tfrac{1}{2}\mu J_0(\mu) = 0,$$

i.e.

$$J_2(\mu) = 0,$$

i.e.

$$\mu = j_{2,l} \quad (l = 1, 2, 3, \ldots). \tag{4.11}$$

The least-damped solution of this type occurs for $l = 1$, when the decay rate $-\beta_i$ is given by (3.15); that is, a larger value than for two of the other modes found. Dr A. Davey (private communication) has obtained numerical results for this mode for a range of finite values of αR. As αR increases from zero, so does $-\beta_i R$, showing that the value given by (3.15) is a minimum value.

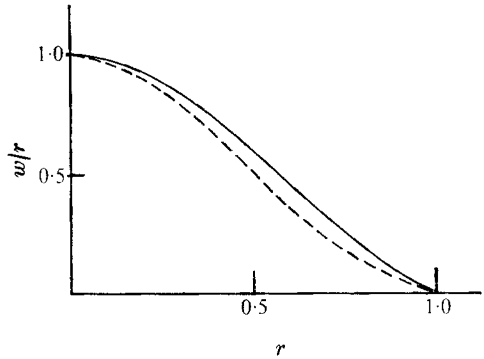

FIGURE 1. The variation of angular velocity w/r about the axis with distance r from the axis for the least-damped temporal (solid line) and spatial (broken line) axisymmetric disturbances (l-modes).

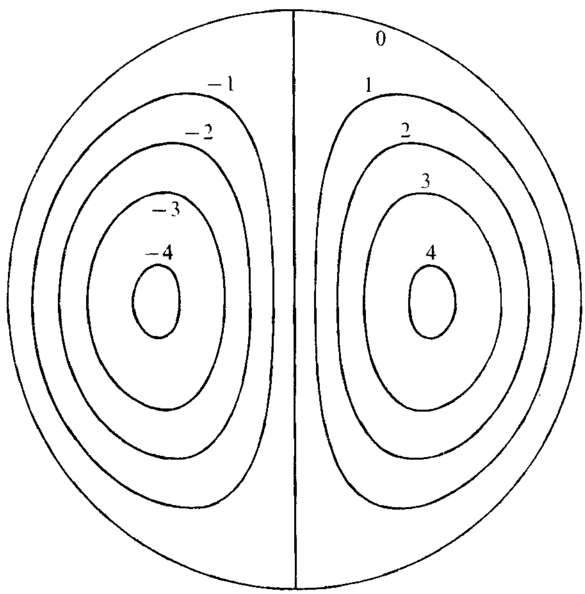

FIGURE 2. Contours of axial velocity u for the non-axisymmetric temporally decaying disturbance (l-mode) with the smallest decay rate. The maximum value of u is 4·16 and the gradient of u at the origin is 13·7.

5. Discussion for temporally decaying disturbances

The significance of the least-damped disturbances is that, if stimulated, they will be the last vestiges of a general decaying disturbance to be seen after a long time. They are *not* particularly relevant to the finite amplitude problem (Davey & Nguyen 1971). Figures 1 and 2 show the properties of the least-damped disturbance. The axisymmetric mode is shown in figure 1. This has motion only in the azimuthal direction, and figure 1 shows how the angular velocity about the axis varies with distance from the axis. From (4.2), the angular velocity is proportional to

$$w/r = 2J_1(\mu r)/\mu r = J_0(\mu r) + J_2(\mu r), \tag{5.1}$$

where $$\mu = j_{1,1}.$$

The least-damped non-axisymmetric mode is given by (3.10). All three velocity components are involved, but v and w are of order α relative to u so may be neglected. Also, since $\alpha \to 0$, the variations in the axial direction are very slow. Thus, at any section, the disturbance will appear to consist only of a perturbation to the axial velocity of the form [see (2.3) and (3.10)]

$$J_1(j_{1,1}r)\cos\theta. \tag{5.2}$$

Contours of this function (multiplied by a convenient constant) are shown in figure 2.

The decay rate of both of these modes is, in non-dimensional terms, $14{\cdot}7R^{-1}$. Since the time scale used is a/U and $R = Ua/\nu$, the dimensional decay rate is

$$14{\cdot}7\nu/a^2. \tag{5.3}$$

This is of the order of the time for viscous effects to diffuse across the pipe, and is independent of U, the velocity of the basic flow. In fact, the limiting equations (3.1)–(3.4) for the l-mode do not depend on the basic flow at all. The above equations also describe the decay of large wavelength disturbances in a circular pipe with no basic flow.

Similar remarks could be made for the case of temporally decaying disturbances to Couette flow between parallel planes. The structure of the l-mode does not depend on the basic flow and the smallest growth rate, in dimensional terms, is

$$4\pi^2\nu/d^2, \tag{5.4}$$

where d is the distance between the two planes which contain the flow. The disturbance velocity for this mode is almost parallel to the walls and has a sinusoidal variation with distance from the walls, with nodes at the two walls and at the mid-plane (i.e. one full wavelength between the two walls). This limiting solution was found by Hopf (1914, pp. 12, 13) and confirmed by Southwell & Chitty (1930) and by Gallagher & Mercer (1962). The numerical results of the latter show (in their table 2) that (5.4) does not quite give the smallest decay rate, values about 3 % lower being obtained at a finite value of αR. It is possible, therefore, that the smallest decay rates for the pipe could be lower than the values obtained for the l-modes by a similar factor. However, Davey has found (see §4) that the l-mode does have the lowest decay rate in at least one case.

It may be appropriate to add a speculative remark here. Attempts to prove stability rigorously often involve comparison with an equation of the same order. The equations for the l-mode may be appropriate for such a comparison because they are associated with a disturbance whose decay rate is close to, or equal to, the minimum value.

Part 2. Spatially decaying disturbances

6. Spatially damped non-axisymmetric *l*-modes

Garg & Rouleau (1972) have made extensive calculations of the properties of spatially decaying disturbances for Reynolds numbers up to 10000 for azimuthal wavenumbers $n = 0$, 1, 2 and 3 and for frequencies β between 0·1 and 1.

For a given mode at a fixed frequency the decay rate α_i decreased as $R \to \infty$. However, for fixed R, the decay rate also decreased as β decreased, and was still decreasing at the smallest value of β considered by Garg & Rouleau. In the particular case of axisymmetric disturbances ($n = 0$), Gill (1965) found by an approximate analysis that the least-damped disturbances were l-modes, i.e. the modes obtained asymptotically in the limit as $\beta R \to 0$ while $R \to \infty$. This result was confirmed by results obtained numerically by Davey & Drazin (1969). The purpose of this part of the paper is to find the l-modes for non-axisymmetric as well as axisymmetric disturbances in the belief that these will be the modes with decay rates equal or close to the smallest possible values.

The problem for spatially damped disturbances is to find the complex eigenvalues α for (2.5)–(2.8) as functions of the real parameters β, n and R. The l-mode corresponds to the limit in which $R \to \infty$, $\beta R \to 0$ but for which αR tends to a finite value. The equations satisfied in this limit for $n \neq 0$ are obtained by applying the limiting process to (2.5)–(2.8) keeping $\alpha u, v, w$ and Rp finite. The resulting equations are

$$u'' + \frac{u'}{r} - \frac{n^2}{r^2}u - i\alpha R(1-r^2)u + 2iRrv = 0, \tag{6.1}$$

$$v'' + \frac{v'}{r} - \frac{n^2+1}{r^2}v - i\alpha R(1-r^2)v - \frac{2n}{r^2}w + iRp' = 0, \tag{6.2}$$

$$w'' + \frac{w'}{r} - \frac{n^2+1}{r^2}w - i\alpha R(1-r^2)w - \frac{2n}{r^2}v - \frac{in}{r}Rp = 0, \tag{6.3}$$

$$\alpha u + v' + \frac{v}{r} + \frac{n}{r}w = 0. \tag{6.4}$$

The boundary condition at $r = 0$ is (3.5).

The solutions satisfying (3.5) can be expanded as power series of the form

$$\alpha u = \sum_{m=1}^{\infty} A_m (\mu r)^{n+2m-2}, \tag{6.5}$$

$$v = \sum_{m=1}^{\infty} B_m (\mu r)^{n+2m-3}, \tag{6.6}$$

$$w = \sum_{m=1}^{\infty} C_m (\mu r)^{n+2m-3}, \tag{6.7}$$

where
$$\mu^2 = -i\alpha R. \tag{6.8}$$

Substitution in (6.1) and (6.4) yield respectively

$$4m(m+n)A_{m+1} + A_m - \epsilon^2 A_{m-1} = 2\epsilon^2 B_m, \tag{6.9}$$

$$(2m+n)B_{m+1} + nC_{m+1} + A_m = 0, \tag{6.10}$$

where
$$\epsilon^2 = 1/\mu^2. \tag{6.11}$$

The remaining two equations (6.2) and (6.3) yield, on elimination of the pressure,

$$4(m-1)(m+n-1)[nB_{m+1} + (2m+n)C_{m+1}] + n(B_m - \epsilon^2 B_{m-1}) + (2m+n-2)(C_m - \epsilon^2 C_{m-1}) = 0. \tag{6.12}$$

A program was written to calculate the coefficients A_m, B_m and C_m and hence, by (6.5)–(6.7), the (complex) values of u, v and w at $r = 1$ for the three cases

$$\left.\begin{aligned} A_1 = 1, \quad B_1 = -C_1 = 0, \quad D = 0,\\ A_1 = 0, \quad B_1 = -C_1 = 1, \quad D = 0,\\ A_1 = 0, \quad B_1 = -C_1 = 0, \quad D = 1,\end{aligned}\right\} \tag{6.13}$$

where $D = B_2 + C_2$. The modulus of the determinant of the three solutions was then calculated as a function of the complex number μ^2. The value which made the determinant zero was found by interpolation, first from a course grid and then from a fine grid of values near the zero as estimated by the first interpolation. The program was checked by first running it with $\epsilon = 0$ in (6.9), (6.10) and (6.12), which gives the temporally damped solutions found analytically in the first section. [This can be seen by comparing (3.1)–(3.4) with (6.1)–(6.4).]

For $n = 1$, the least-damped disturbance found in this way had α given by

$$\alpha R = 10{\cdot}82 + 28{\cdot}68i. \tag{6.14}$$

For $n = 2$, the least-damped disturbance had α given by

$$\alpha R = 17{\cdot}77 + 49{\cdot}88i. \tag{6.15}$$

For $n = 3$, the smallest damping rate had $\alpha_i R > 80$.

7. Spatially damped axisymmetric *l*-modes

The determination of the limiting solution in the axisymmetric case follows the same lines as for the temporally damped disturbances. There are two independent solutions, one involving w alone ($u = v = 0$) and one involving only u, v and p ($w = 0$).

First consider the equation for w. In the limit, this becomes (6.3) with $n = 0$. The solution has the form (6.7) with $C_1 = 0$, and the coefficients C_m can be calculated from (6.12) for $n = 0$. In this case, values of μ^2 are real and the smallest damping rate α_i is given by

$$\alpha_i R = 21{\cdot}38. \tag{7.1}$$

The power-series solution is

$$\begin{aligned} w = r - 2{\cdot}673r^3 + 3{\cdot}272r^5 - 2{\cdot}648r^7 + 1{\cdot}582r^9 - 0{\cdot}754r^{11} + 0{\cdot}297r^{13}\\ - 0{\cdot}100r^{15} + 0{\cdot}030r^{17} - 0{\cdot}008r^{19} + 0{\cdot}002r^{21}. \end{aligned} \tag{7.2}$$

This solution is shown in figure 1 for comparison with the corresponding temporally damped mode.

Now consider the equations for u, v and p. One limiting solution satisfies (6.1), (6.2) and (6.4) and is given by (6.5) and (6.6) with $A_1 = 1$ and $B_1 = 0$. This solution was found using (6.9) and (6.10). The other solution is the inviscid solution for which p is of order $1/\alpha$ in the limit. In this limit, (2.5), (2.6) and (2.8) become

$$\alpha(1-r^2)\,u - 2rv = -\alpha p, \tag{7.3}$$

$$0 = p', \tag{7.4}$$

$$\alpha u + v' + v/r = 0, \tag{7.5}$$

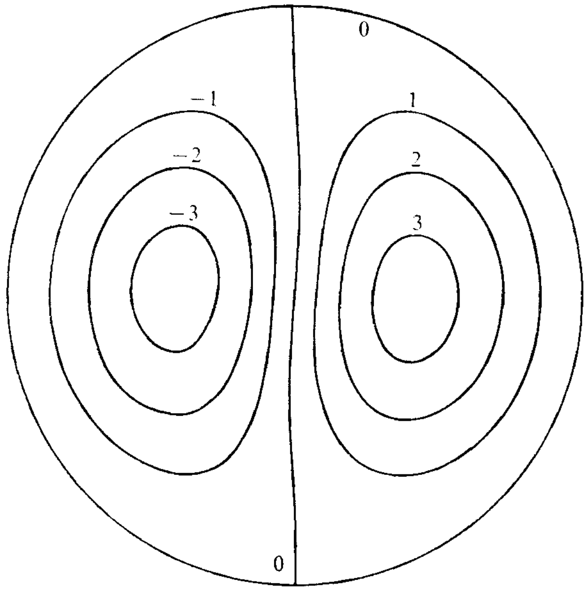

FIGURE 3. Contours of axial velocity u for the non-axisymmetric spatially decaying disturbance (l-mode) with the smallest decay rate. The maximum value of u is 3·46 and the gradient of u at the origin is 13·7 (i.e. the same as in figure 2).

and the solution is

$$\alpha u = -1, \quad v = \tfrac{1}{2}r, \quad \alpha p = -1. \tag{7.6}$$

The values of μ [see (6.8)] which allow the no-slip condition at $r = 1$ to be satisfied are real, and were calculated by Gill (1965). The least-damped disturbance of this type has damping rate α_i given by

$$\alpha_i R = 32{\cdot}08. \tag{7.7}$$

Dr A. Davey (private communication) has obtained numerical results for this case for finite values of βR. $\alpha_i R$ increases as βR increases from zero, confirming that (7.7) gives the lowest damping rate in this case.

8. Discussion

It is more common for disturbances in a pipe to decay in space rather than in time, so perhaps the spatially damped modes are useful in describing the last vestiges of a decaying disturbance. The least-damped mode is the axisymmetric one with only an azimuthal component of velocity, and this is depicted in figure 1.

The least-damped non-axisymmetric mode decays at a rate which is 34 % greater. All three velocity components are involved, but v and w are of order α relative to u and so may be neglected. Also, since $\alpha \to 0$, the variations in the axial direction are very slow. Thus, at any section, the disturbance will appear to consist only of a perturbation to the axial velocity of the form [see (2.3)]

$$u_r(r)\cos\theta - u_i(r)\sin\theta, \tag{8.1}$$

where u_r and u_i are the real and imaginary parts of $u(r)$. Contours of this function for the least-damped mode are shown in figure 3.

The smallest decay rate (i.e. for the axisymmetric mode) is, in non-dimensional terms, $21{\cdot}4R^{-1}$. Since the space scale used is a and $R = Ua/\nu$, the dimensional decay rate is

$$21{\cdot}4\nu/Ua^2. \tag{8.2}$$

This is of the order of the distance for viscous effects to diffuse across the pipe when the advection velocity is of order U. It corresponds, for instance, to decay with time at the rate given by (5.3) when there is advection at a uniform velocity of $0{\cdot}69U$.

Thus, to summarize, modes have been found in a certain limit which have decay rates given by (5.3) for decay with time, or by (8.2) for decay with distance. These are thought to have, for a given large Reynolds number, the smallest decay rates possible.

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